Embedding Obstructions for the Dihedral, Semidihedral, and Quaternion 2-Groups

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For each of the dihedral, semidihedral, and quaternion 2-groups, we represent the obstructions to certain Brauer problems as tensor products of quaternion algebras. Then we reduce various embedding problems with cyclic 2-kernels into two Brauer problems, thus finding the obstructions in some specific cases.

1. INTRODUCTION

Let $K/k$ be a Galois extension with Galois group $F$, and let

\[(*)\quad 1 \rightarrow A \rightarrow G \rightarrow F \rightarrow 1\]

be a finite group extension. The embedding problem $(K/k, G, A)$ then consists of determining whether there exists a Galois extension $L/k$ such that $K \subset L$, $G \cong \text{Gal}(L/k)$, and the homomorphism of restriction to $K$ of the automorphisms from $G$ coincides with $\pi$. The group $A$ is called the kernel of the embedding problem. If there exists a Galois algebra with the aforementioned properties, then we also talk about “weak” solvability. Given that $A$ is contained in the Frattini subgroup of $G$, i.e., $\text{rank}(G) = \text{rank}(F)$, the two terms are equivalent.

Let $A$ be a cyclic group of order $m$, let $\zeta \in K$ be a primitive $m$th root of unity, and denote $\mu_m = \langle \zeta \rangle \subset K^*$. If $F$ acts on $A$ and $\mu_m$ in the same way, then the embedding problem $(K/k, G, A)$ is called a Brauer problem. We can identify $A$ with $\mu_m$ and denote by $c$ the 2-coclass of the extension $(*)$ in $H^2(F, \mu_m)$. It is well known (see [Mi2, ILF]) that $(K/k, G, A)$ is
solvable if and only if $c$ maps to 1 under the map $H^2(F, \mu_m) \to H^2(F, K^*)$, induced by the inclusion $\mu_m \subset K^*$. In this way we consider $c$ as an element of the relative Brauer group $Br(K/k) \cong H^2(F, K^*)$. The element $c$ is called the (embedding) obstruction to the embedding problem $(K/k, G, A)$. Then $c = 1 \in Br(k)$ gives us the condition for solvability.

Let $k$ be of characteristic $\neq 2$ and let $m = 2^n$; i.e., let $\zeta \in K$ be a primitive $2^n$th root of unity and $A \cong C_{2^n}$. Then we can split the algebra representing the obstruction to the Brauer problem into a tensor product of quaternion algebras and matrix algebras. Namely, for the solvability of the embedding problem $(K/k, G, \mu_{2^n})$ it is necessary the solvability of the associated problem $(K/k, G/C_{2^n}, \mu_{2^n-1})$. The latter has an obstruction $c^2 \in Br(K/k)$. Given $c^2 = 1 \in Br(k)$ by Merkurjev theorem [Me] $c$ can be split into a product of quaternion classes. All needed information about quaternion algebras and Brauer groups can be found in, for example, [La].

If the embedding problem $(K/k, G, C_{2^n})$ is not Brauer but $\zeta + \zeta^{-1} \in k$ and $i(\zeta - \zeta^{-1}) \in k$, then we can reduce it to two Brauer problems. We do this in Section 2.

In Section 3 we apply Theorem 2.1 and Corollary 2.2 to find the obstruction to the embedding problem given by a $D_8$ extension $K/k$ and the group extension

$$1 \to C_{2^n} \to G \to D_8 \to 1,$$

where $G$ is isomorphic to either the dihedral, the semidihedral, or the quaternion groups of order $2^{n+3}$ ($n \geq 1$). We investigate four such embedding problems in all possible cases according to the location of $\zeta$ in $K$.

The representation of obstructions as products of quaternion classes is a difficult problem even if $G$ is a group of order 16 (see [Le1, GSS]). We obtain in the meantime a single obstruction with two parameters for the dihedral, semidihedral, and quaternion groups of order 16. In Theorem 3.2, given that $\zeta + \zeta^{-1} \in k$ and $i(\zeta - \zeta^{-1}) \in k$, we again find a single obstruction that is valid for the three 2-groups instead of investigating each Brauer problem separately.

2. EMBEDDING PROBLEMS WITH CYCLIC 2-KERNELS

Let $K/k$ be a Galois extension with Galois group $F$, and consider the embedding problem given by $K/k$ and the finite group extension

$$\begin{align*}
1 & \to A \to G \to F \to 1.
\end{align*}$$

(2.1)

Let the kernel $A$ be Abelian of order $n$, and let a primitive $n$th root of unity $\zeta$ be in $K$. Then we can form the character group $\hat{A} = \{\chi: A \to K^*\}$,
where \( \chi \) is a homomorphism of \( A \) to the group of roots of unity contained in \( K \). We denote the action of \( F \) on \( A \) by

\[
\sigma a = \tilde{\sigma}^{-1} a \tilde{\sigma}, \quad \sigma \in F, a \in A,
\]

where \( \tilde{\sigma} \) is a preimage of \( \sigma \) in \( G \). We denote the action of \( F \) on \( K \) by \( \sigma x \) for \( x \in K \), and introduce the action of \( F \) on \( \hat{A} \) by

\[
\chi^{\sigma}(a) = \chi(\sigma^{-1} a), \quad \sigma \in F, a \in A.
\]

An embedding problem is called \textit{Brauer} if \( \chi^{\sigma} = \chi \) for all \( \chi \in \hat{A} \) and \( \sigma \in F \). By [ILF, Theorem 3.2], the compatibility condition is necessary and sufficient for solvability of the Brauer problem with Abelian kernel.

We introduce the following notations: \( F_{\chi} = \{ \sigma \in F, \chi^{\sigma} = \chi \} \), the subgroup of \( F \) which acts on certain character \( \chi \in \hat{A} \) trivially; \( B_{\chi} = \text{Ker} \chi \), the kernel of \( \chi \); and \( A_{\chi} = A/B_{\chi} \), \( H_{\chi} = \pi^{-1}(F_{\chi}) \), \( G_{\chi} = H_{\chi}/B_{\chi} \) and \( K_{\chi} \), the fixed field of \( F_{\chi} \). Then the compatibility condition for the embedding problem \( (K/k, G, A) \) holds if and only if the associated problems \( (K/K_{\chi}, G_{\chi}, A_{\chi}) \) related to the group extensions

\[
(2.2) \quad 1 \to A_{\chi} \to G_{\chi} \to F_{\chi} \to 1
\]

are solvable for all characters \( \chi \). It is clear that \( A_{\chi} \) is a cyclic group and \( \chi \) is a generator of the character group of \( A_{\chi} \).

In cohomological terms, the group \( \mu_{n} \) of \( n \)th roots of unity is embedded in the multiplicative group \( K^{*} \) of the field \( K \). Then the character \( \chi \) induces a homomorphism \( \overline{\chi}: H^{2}(F_{\chi}, A_{\chi}) \to H^{2}(F_{\chi}, K^{*}) \), and the compatibility condition can be stated as \( \overline{\chi}(c_{\chi}) = 1 \) for all \( \chi \in \hat{A} \), where \( c_{\chi} \) is the 2-coclass of \( (2.2) \) in \( H^{2}(F_{\chi}, A_{\chi}) \).

In fact, we need not consider all of these problems. It suffices to consider the problems \( (K/K_{\chi}, G_{\chi}, A_{\chi}) \), where \( \chi \) runs through a set of representatives of the conjugate classes in \( \hat{A} \), considered as an \( F \)-module. In particular, for a Brauer problem with cyclic kernel, all characters are powers of a certain \( \chi, F_{\chi} = F \) and \( K_{\chi} = k \), so the compatibility condition obtains the form \( \overline{\chi}(c) = 1 \), where \( c \) is the class of \( (2.1) \) in \( H^{2}(F, A) \) and \( \overline{\chi}: H^{2}(F, A) \to H^{2}(F, K^{*}) \).

Now let \( A = C_{4} \) be generated by an element \( a \), and let \( i \in K \) be a primitive fourth root of unity. Then \( \hat{A} \) is generated by an element \( \chi \) such that \( \chi(a) = i \). If the embedding problem \( (K/k, G, C_{4}) \) is not Brauer, then there exists \( \kappa \in F \) such that \( \chi^{\kappa} = \chi^{-1} \), so \( N = F_{\chi} \) is a subgroup of \( F \) of index 2. Hence \( N \) is the maximal subgroup of \( F \) which acts on \( C_{4} \) and \( \mu_{4} \) in the same way. We see that \( \text{Ker} \chi = \{1\}, A_{\chi} = C_{4}, G_{\chi} = H_{\chi} = \pi^{-1}(N), \text{Ker} \chi^{2} = \{1, a^{2}\} \cong C_{2}, A_{\chi}^{2} \cong C_{2}, F_{\chi}^{2} \cong F, \) and \( G_{\chi}^{2} \cong G/C_{2} \).

The conjugate classes in \( \hat{A} \) are \( \{1\}, \{\chi, \chi^{-1}\}, \) and \( \{\chi^{2}\} \). Therefore, the
compatibility condition holds if and only if the associated problems \((K/k_1, \pi^{-1}(N), C_4)\) and \((K/k, G/C_2, C_2)\), related to the group extensions

\[
1 \rightarrow \mu_4 \rightarrow \pi^{-1}(N) \rightarrow \pi \rightarrow 1,
\]

where \(k_1 = K_\chi\) is the fixed field of \(N\) and

\[
1 \rightarrow C_2 \rightarrow G/C_2 \rightarrow F \rightarrow 1
\]

are solvable. By [ILF, Section 4] the compatibility condition for embedding problems with cyclic kernel of order 4 is also sufficient for solvability (see also [MZ, Corollary 3.3]). We define homomorphisms \(e, f, g\) from \(F\) in \(\{+1, -1\}\) by \(\sigma a = a^\sigma, \sigma i = i^\sigma, \) and \(g_\sigma = e_{\sigma}f_\sigma\). Then \(N = \{\sigma \in F, g_\sigma = 1\}\), so we obtain from another point of view Ledet's result [Le2, Theorem 1.1].

Now let \(A = C_2, n \geq 2\), be generated by an element \(\alpha\), let \(K\) contain a primitive \(2^n\)th root of unity \(\zeta\), and let \(\chi: C_2 \rightarrow K^\star\) be a generator of \(\bar{C}_2^\star\). Clearly, for an odd \(m\) we have \(F_\chi = F_{\chi^m}\) and \(\text{Ker} \chi = \text{Ker} \chi^m = \{1\}\). Also let \(F_\chi\) be of index 2 in \(F\). Then from \(F_\chi \subset F_{\chi^m}\), it follows that \(F_\chi = F_{\chi^m}\) or \(F = F_{\chi^m}\).

If \(F_\chi = F_{\chi^m}\), then we obtain the Brauer problem \((K/K_\chi, \pi^{-1}(F_\chi)/B_{2m}, C_2^\star/B_{2m})\), which is an associated problem of the first kind to the problem \((K/K_\chi, \pi^{-1}(F_\chi), C_2)\). Here, for abuse of notation, \(B_{2m} \cong B_{2^m}\).

If \(F = F_{\chi^m}\), then we obtain the Brauer problem \((K/k, G/B_{2m}, C_2^\star/B_{2m})\). From this type of problem, we need to consider only the one with the “biggest” kernel. Namely, let \(\chi^\sigma = \chi^m\), and \(m_\sigma \in \mathbb{N}\), and since \(F_\chi\) is of index 2, we have \(m_\sigma \in \{1, l\}\), where \(l\) is odd such that \(l^2 \equiv 1 \pmod{2^n}\). Hence \((\chi^m)^\sigma = \chi^{2m}\) for all \(\sigma \in F\) if and only if \(2ml \equiv 2m \pmod{2^n}\), i.e., \(ml \equiv m \pmod{2^{n-1}}\). Now let \(m_0\) be the minimal natural number such that \(1 \leq m_0 \leq 2^{n-2}\) and \(m_0/l \equiv m_0 \pmod{2^{n-1}}\). Thus, if \(m\) is such that \(F = F_{\chi^m}\), then \(B_{2m_0} \subset B_{2m}\), so we get the isomorphisms \((C_2^\star/B_{2m_0})/(B_{2m}/B_{2m_0}) \cong C_2^\star/B_{2m}\) and \((G/B_{2m_0})/(B_{2m}/B_{2m_0}) \cong G/B_{2m}\). Therefore, the embedding problem \((K/k, G/B_{2m}, C_2^\star/B_{2m})\) is an associated problem of the first kind to the problem \((K/k, G/B_{2m_0}, C_2^\star/B_{2m_0})\). In this way the compatibility condition of the problem \((K/k, G, C_2^\star)\) is equivalent to the solvability of the two problems \((K/K_\chi, \pi^{-1}(F_\chi), C_2)\) and \((K/k, G/B_{2m_0}, C_2^\star/B_{2m_0})\). There are two important cases:

1. If \(l \equiv 1 \pmod{2^{n-1}}\), then \(m_0 = 1\) and \(B_2 \cong C_2\), and so the latter embedding problem is \((K/k, G/C_2, C_2^\star)\).

2. If \(l \equiv -1 \pmod{4}\), then \(m_0 = 2^{n-2}\) and \(B_{2^{n-1}} \cong C_2^{2^{n-1}}\), and so the latter embedding problem is \((K/k, G/C_2^{2^{n-1}}, C_2)\).

In this way we once again obtain the following results, which are proved explicitly in [Mi2].
\textbf{Theorem 2.1.} Let $K/k$ be a finite Galois extension with Galois group $F$, and let $\zeta \in K$ be a primitive $2^n$th root of unity ($n > 1$). Consider the group extension 

\begin{equation}
1 \rightarrow C_{2^2} \rightarrow G \rightarrow F \rightarrow 1
\end{equation}

such that $e_{\sigma}, f_{\sigma} \in \{+1, -1\}$ for all $\sigma \in F$. Let $k_1$ be the fixed field of $N = \K_{\text{er}}$. Then the embedding problem $(K/k, G, C_{2^n})$ is solvable if and only if the embedding problems $(K/k, \pi^{-1}(N), \mu_{2^n})$ and $(K/k, G/C_{2^n-1}, \mu_2)$ are solvable.

\textbf{Corollary 2.2.} Let $K/k$ be a finite Galois extension with Galois group $F$, and let $\zeta$ be a primitive $2^n$th root of unity ($n > 1$) such that $\zeta + \zeta^{-1} \in k$, $i(\zeta - \zeta^{-1}) \in k$ and $i \notin K$. Let

\begin{equation}
1 \rightarrow C_{2^2} \rightarrow G \rightarrow F \rightarrow 1
\end{equation}

be a group extension. Extend the elements $\sigma \in F$ to $K(i)$ by $\sigma i = i$, and let $\kappa$ be the generator of $\text{Gal}(K(i)/K)$. Let $k(\sqrt{b})$ be the fixed field of $N = \K_{\text{er}}$ and $k_1 = k(i\sqrt{b})$. Then $\text{Gal}(K(i)/k_1) \cong F$, and the embedding problem $(K/k, G, C_{2^n})$ is solvable if and only if the embedding problems $(K(i)/k_1, G, \mu_{2^n})$ and $(K/k, G/C_{2^n-1}, \mu_2)$ are solvable.

The foregoing results imply that the embedding problem $(K/k, G, C_{2^n})$ has two obstructions corresponding to the two reduced Brauer problems. In the following section we decompose each obstruction into a product of quaternion classes.

For $a, b \in k^*$, we denote by $(a, b)$ the equivalence class in $\text{Br}(k)$ of the quaternion algebra generated over $k$ by elements $i$ and $j$ with relations $i^2 = a, j^2 = b$ and $ij = -ji$. We note that when elements $i$ and $j \neq 0$ with relations $i^2 = a^2, j^2 = 0$, and $ij = -ji$ show up in a centraliser (see Theorem 3.2 below), they demonstrate that the centraliser is split, even though they do not generate it.

\section{3. The Dihedral, Semidihedral, and Quaternion Groups}

In this section we investigate embedding problems involving the dihedral ($D_{2^n}$), semidihedral ($SD_{2^n}$), and quaternion ($Q_{2^n}$) groups of order $2^n, n \geq 4$. Their presentations are as follows:

\begin{align*}
D_{2^n} & \cong \langle \sigma, \tau \mid \sigma^{2^n-1} = \tau^2 = 1, \tau \sigma = \sigma^{-1} \tau \rangle \\
SD_{2^n} & \cong \langle \sigma, \tau \mid \sigma^{2^n-1} = \tau^2 = 1, \tau \sigma = \sigma^{2^{n-1}-1} \tau \rangle \\
Q_{2^n} & \cong \langle \sigma, \tau \mid \sigma^{2^n-1} = 1, \tau^2 = \sigma^{2^{n-1}}, \tau \sigma = \sigma^{-1} \tau \rangle
\end{align*}
First, consider the case where \( n = 4 \). Let \( K/k = k(\sqrt[8]{r(\alpha + \beta \sqrt{a})}, \sqrt{b})/k \) be a \( D_8 \) extension, where \( a \) and \( b \) are quadratically independent, \( r \in k^* \), and \( \alpha, \beta \in k \), such that \( \alpha^2 - a\beta^2 = ab \). Denote \( \varphi = \sqrt{r(\alpha + \beta \sqrt{a})} \) and \( \psi = \sqrt{r(\alpha - \beta \sqrt{a})} \). Then \( \varphi \psi = r\sqrt{ab} \), and \( D_8 \) is generated by elements \( \sigma \) and \( \tau \) such that
\[
\sigma: \varphi \mapsto \psi, \psi \mapsto -\varphi, \sqrt{b} \mapsto \sqrt{b}
\]
\[
\tau: \varphi \mapsto \varphi, \psi \mapsto -\psi, \sqrt{b} \mapsto -\sqrt{b}.
\]
Now consider the group extension
\[
1 \to C_2 = \{\pm 1\} \to G \xrightarrow{i_{\varphi}} D_8 \to 1,
\]
where \( s \) and \( t \) are preimages in \( G \) of \( \sigma \) and \( \tau \), respectively, such that \( s^4 = -1 \), \( t^2 = e_1 \), and \( ts = e_2 s^3 t \) for \( e_1 = (-1)^m \), \( e_2 = (-1)^m \), and \( m_1, m_2 \in \{0, 1\} \).

The crossed product algebra \( \Gamma = (K, D_8, -1) \), corresponding to the extension, contains the following three quaternion subalgebras:
\[
Q_1 : i_1 = t, \quad j_1 = \sqrt{b}
\]
\[
Q_2 : i_2 = (s + s^3)\sqrt{b}^{-m_2}, \quad j_2 = \sqrt{a}
\]
\[
Q_3 : i_3 = s^2\sqrt{b}, \quad j_3 = (\varphi + \psi s)\sqrt{a}.
\]
We see that \( i_1^2 = (-1)^m_1 \), \( i_2^2 = b \), \( i_3^2 = -2b^{-m_2} \), \( j_2 = a \), \( j_3 = -b \), and \( j_2^2 = 2\varphi a \). Since \( Q_1 \), \( Q_2 \), and \( Q_3 \) centralize each other, we get
\[
[\Gamma] = [Q_1][Q_2][Q_3] = ((-1)^m_1, b)(-2b^{-m_2}, a)(-b, 2\varphi a) \in \text{Br}(k).
\]
Thus we get the following theorem.

**Theorem 3.1.** The obstructions to the embedding problem \( (K/k, G, C_2) \) are as follows:

1. \( m_1 = 0, m_2 = 1 \) (\( G = D_{16} \)) : \( (a, 2)(-b, 2\varphi a) \in \text{Br}(k) \)
2. \( m_1 = m_2 = 1 \) (\( G = Q_{16} \)) : \( (a, 2)(b, b)(-b, 2\varphi a) \in \text{Br}(k) \)
3. \( m_1 = m_2 = 0 \) (\( G = SD_{16} \)) : \( (a, -2)(-b, 2\varphi a) \in \text{Br}(k) \)
4. \( m_1 = 1, m_2 = 0 \) (\( G = SD_{16} \)) : \( (a, -2)(b, b)(-b, 2\varphi a) \in \text{Br}(k) \).

Note that we have two distinct obstructions for \( SD_{16} \), since the two corresponding group extensions are nonequivalent. A thorough discussion of the obstructions for the groups of order 16 can be found in [GSS, Ki, Le1].

Now, let \( K/k \) be a \( D_8 \) extension and let \( \zeta \in K \) be a primitive \( 2^n \)th root of unity such that \( \zeta \not\in k \), \( \zeta + \zeta^{-1} \in k \), and \( i(\zeta - \zeta^{-1}) \in k \).
Then $K/k = k(\sqrt{a}, i)$ for some $a \in k \setminus k^2$, and $D_8$ is generated by elements $\sigma$ and $\tau$, given by

$$
\sigma: \sqrt{a} \mapsto i\sqrt{a}, \ i \mapsto i; \ \tau: \sqrt{a} \mapsto -\sqrt{a}, \ i \mapsto -i
$$

(in particular, $\sigma(\xi) = \zeta$ and $\tau(\xi) = \zeta^{-1}$).

We now turn our attention to the case when $G$ is a group generated by elements $s$ and $t$, such that $s$ is of order $2^{n+2}$, $t^2 = e_1$ and $ts = e_2 s^{-1} t$, where $e_1^2 = e_2^2 = 1$. Since $ts^4 = s^{-3} t$, we can put $s^4 = \xi$, and get the group extension

$$
1 \rightarrow \mu_{2^n} \rightarrow G \rightarrow D_8 \rightarrow 1,
$$

where we identify the cyclic group $\langle s^4 \rangle$ with the group of $2^n$th roots of unity $\mu_{2^n}$. Therefore, we have $s^4 = \zeta, t^2 = e_1$, and $ts = e_2 \zeta^{-1} s^2 t$, where $e_1, e_2 \in \{+1, -1\}$. The group $G$ has an element of order $2^{n+2}$, and hence $G$ is isomorphic either to the dihedral, semidihedral, or quaternion group of order $2^{n+3}$. Our main result of this section is calculation of the obstruction to the embedding problem $(K/k, G, \mu_{2^n})$ in the following theorem.

**Theorem 3.2.** For the solvability of the embedding problem $(K/k, G, \mu_{2^n})$ for $n \geq 1$, it is necessary that there exists $\alpha_1 \in k^*$ and $\beta_1 \in k$, such that $\alpha_1^2 + a \beta_1^2 = 2 - \xi - \xi^{-1}$. In that case the obstruction is

$$
(-1, e_1)(2 + \xi + \xi^{-1}, \alpha_1 \beta_1) \left( a, e_2 a_1 \left( 2 \alpha_1 - \frac{\xi - \xi^{-1}}{i} \right) \right) \in \text{Br}(k).
$$

**Proof.** We proceed by induction. For $n = 1$, we have $\xi = -1$ and let $\alpha_1 = 2, \beta_1 = 0 : \alpha_1^2 + a \beta_1^2 = 2 - \xi - \xi^{-1} = 4$. Then we get the obstruction $(-1, e_1)(a, 2 e_2) \in \text{Br}(k)$, which can also be obtained from Theorem 3.1 for $b = -1$. Now let the embedding problems for $n - 1$ be solvable. In particular, the associated problem $(K/k, D_{2^{n+2}}, \mu_{2^{n-1}})$ is solvable (here $e_1 = e_2 = 1$). Then $\xi^2$ is a primitive $2^n$th root of unity and $2 - \xi^2 - \xi^{-2} = \left( \frac{\xi - \xi^{-1}}{i} \right)^2$, so we can let $\alpha_1 = \frac{\xi - \xi^{-1}}{i}$ and $\beta_1 = 0$. Thus by the induction assumption, the obstruction to $(K/k, D_{2^{n+2}}, \mu_{2^{n-1}})$ is

$$
((\xi + \xi^{-1})^2, 0) \left( a, \frac{\xi - \xi^{-1}}{i} \left( 2 \frac{\xi - \xi^{-1}}{i} - \frac{\xi^2 - \xi^{-2}}{i} \right) \right)
$$

$$
= \left( a, \frac{\xi - \xi^{-1}}{i} \left( 2 \frac{\xi - \xi^{-1}}{i} - \frac{\xi - \xi^{-1}}{i} (\xi + \xi^{-1}) \right) \right)
$$

$$
= \left( a, \left( \frac{\xi - \xi^{-1}}{i} \right)^2 \left( 2 - \xi - \xi^{-1} \right) \right) = (a, 2 - \xi - \xi^{-1}) \in \text{Br}(k).
$$
Further,

\[ (2 - \zeta - \zeta^{-1})(2 + \zeta + \zeta^{-1}) = 4 - (\zeta + \zeta^{-1})^2 \]

\[ = 2 - \zeta^2 - \zeta^{-2} = \left( \frac{\zeta - \zeta^{-1}}{i} \right)^2 \in k^2 \]

and

\[ \left( 1 + \frac{\zeta + \zeta^{-1}}{2} \right)^2 + \left( \frac{\zeta - \zeta^{-1}}{2i} \right)^2 \]

\[ = 1 + \zeta + \zeta^{-1} + \frac{\zeta^2 + \zeta^{-2}}{4} + 1 - \frac{\zeta^2 + \zeta^{-2}}{4} + \frac{1}{2} \]

\[ = 2 + \zeta + \zeta^{-1}. \]

Hence both \(2 + \zeta + \zeta^{-1}\) and \(2 - \zeta - \zeta^{-1}\) are sums of two squares in \(k\). Thus we obtain that \((-a, 2 - \zeta - \zeta^{-1}) = 1 \in Br(k)\) (or, equivalently, \((-a, 2 + \zeta + \zeta^{-1}) = 1 \in Br(k)\)) is necessary for solvability of the embedding problem \((K/k, G, \mu_{2n})\) for \(n > 1\).

Now let \(\alpha_2 \in k^*\) and \(\beta_2 \in k\) be such that \(\alpha_2^2 + a\beta_2^2 = 2 + \zeta + \zeta^{-1}\). The connection between \(\alpha_2, \beta_2\) and \(\alpha_1, \beta_1\) is given by

\[ \alpha_1^2 + a\beta_1^2 = 2 - \zeta - \zeta^{-1} = \frac{2 - \zeta^2 - \zeta^{-2}}{2 + \zeta + \zeta^{-1}} \]

\[ = (2 + \zeta + \zeta^{-1})\left( \frac{\zeta - \zeta^{-1}}{i(2 + \zeta + \zeta^{-1})} \right)^2. \]

We let \(\gamma = \frac{\zeta - \zeta^{-1}}{i(2 + \zeta + \zeta^{-1})} \in k, \alpha_2 = \frac{\alpha_1}{\gamma}\), and \(\beta_2 = \frac{\beta_1}{\gamma}\) and get \(\alpha_2^2 + a\beta_2^2 = 2 + \zeta + \zeta^{-1}\).

Let \(\Gamma\) be the algebra representing the obstruction. Then \(\Gamma\) is generated by two elements \(u\) and \(v\) over \(K\) such that \(u^4 = \zeta, v^2 = \epsilon_1, vu = \epsilon_2 u^{-1}v = \epsilon_2 \zeta^{-1} u^3 v, ux = \sigma(x)u, \) and \(vx = \tau(x)v\) for \(x \in K\). Then \(\Gamma\) contains the following three quaternion subalgebras:

- \(Q_1: i_1 = i,\)
  \(j_1 = v\)
- \(Q_2: i_2 = (1 + \zeta^{-1})u^2,\)
  \(j_2 = \sqrt{a}(\alpha_2 + \beta_2 \sqrt{a} + \epsilon_2 (1 + \zeta^{-1})u^2)\)
- \(Q_3: i_3 = \sqrt{a},\)
  \(j_3 = -(1 + i)(1 + \zeta^{-1}) + \alpha_2 (1 + i) + (1 - i) \beta_2 \sqrt{a} u + \epsilon_3 \zeta^{-1} [-(1 - i)(1 + \zeta^{-1}) + \alpha_2 (1 - i) + (1 + i) \beta_2 \sqrt{a} u].\)

Calculations show that \(i_1^2 = -1, j_1^2 = \epsilon_1, i_2^2 = 2 + \zeta + \zeta^{-1}, j_2^2 = 2 \alpha_2 \beta_2 a, i_3^2 = a, \) and \(j_3^2 = \epsilon_2^4 \alpha_2 (2 \alpha_2 - 2 - \zeta - \zeta^{-1}).\) Also, \(i_s j_s = -j_s i_s, 1 \leq s \leq 3,\)
and the generators of each algebra pairwise commute with the generators of the other two. We are forced to omit the monstrous verification, however.

Thus finally we obtain

\[
[\Gamma] = [Q_1][Q_2][Q_3]
\]

\[
= (-1, e_1)(2 + \zeta + \zeta^{-1}, \alpha_2 \beta_2) (a, e_2 \alpha_2 (2\alpha_2 - 2 - \zeta - \zeta^{-1}))
\]

\[
= (-1, e_1)(2 + \zeta + \zeta^{-1}, \alpha_1 \beta_1)\left(a, e_2 \alpha_1 \left(\frac{2\alpha_1}{\gamma} - 2 - \zeta - \zeta^{-1}\right)\right)
\]

\[
= (-1, e_1)(2 + \zeta + \zeta^{-1}, \alpha_1 \beta_1)\left(a, e_2 \alpha_1 \left(\frac{\zeta - \zeta^{-1}}{i}\right)\right) \in \text{Br}(k). \quad \blacksquare
\]

**Remark 3.1.** If it happens that for \( n \geq 3 \) we have \( \alpha_1 = 0 \) and \( a\beta_1^2 = 2 - \zeta - \zeta^{-1} \), then we can put \( \alpha_1' = \frac{2}{3+\zeta+\zeta^{-1}}(\zeta-\zeta^{-1}) \) and \( \beta_1' = \frac{1+\zeta+\zeta^{-1}}{3+\zeta+\zeta^{-1}}\beta_1 \), and hence \( \alpha_1'^2 + a\beta_1'^2 = 2 - \zeta - \zeta^{-1} \). For \( n = 2 \), this works well if \( k \) has characteristic \( \neq 3 \), then we simply have \( \alpha_1' = \frac{\zeta^2}{3} \) and \( \beta_1' = \frac{1}{3}\beta_1 \). If \( k \) has characteristic 3, then we can put \( \alpha_1' = a - 1/a \) and \( \beta_1' = (a + 1/a)\beta_1 \), so \( \alpha_1'^2 + a\beta_1'^2 = 2 \).

**Remark 3.2.** For \( n \geq 3 \), we have that \( \zeta^2 + \zeta^{-2} \in k \) and \( 2 \in k^2 \). From the proof we also get that \( (a, 2 - \zeta^2 - \zeta^{-2}) = 1 \in \text{Br}(k), \quad 0 \leq s \leq n - 1 \), is necessary for solvability of the embedding problem.

Helping our consideration is the following lemma, obtained in [Mi2].

**Lemma 3.3.** Let \( \zeta \in k \) be a primitive \( 2^n \)th root of unity \( (n \geq 1) \), and let \( i \in k \). For the embedding problem given by a \( C_4 \) extension \( k(\sqrt{a})/k \) and the group extension

\[
1 \to \mu_{2^n} \hookrightarrow C_{2^{n+1}} \to C_4 \to 1
\]

(3.2)

to be solvable, it is necessary that there exist \( \alpha', \beta' \in k \), \( \alpha' \neq 0 \), such that \( \alpha'^2 - a\beta'^2 = \zeta \). In that case, the obstruction is \( (a, \alpha')(\zeta, a\beta') \in \text{Br}(k) \).

Now consider the group extension

\[
1 \to C_{2^n} \to G \to D_8 \to 1,
\]

(3.3)

where \( G \) is generated by elements \( x \) of order \( 2^{n+2} \) and \( y \). We then have four non-equivalent group extensions lifting an element of order 4 to one of order \( 2^{n+2} \):

(3.4a) \[
1 \to C_{2^n} \to D_{2^{n+3}} \to D_8 \to 1
\]

(3.4b) \[
1 \to C_{2^n} \to Q_{2^{n+3}} \to D_8 \to 1
\]
\[ (3.4c) \quad 1 \to C_{2^a} \to SD_{2^{a+3}} \to D_8 \to 1 \]

\[ (3.4d) \quad 1 \to C_{2^a} \to SD_{2^{a+1}} \to D_8 \to 1. \]

Assume again that \( \xi + \xi^{-1} \in k \) and \( i(\xi - \xi^{-1}) \in k \), so the location of \( \xi \) in \( K/k \) is determined by the location of \( i \). Recall that \( K/k = k(\sqrt{r(\alpha + \beta\sqrt{a})}, \sqrt{b})/k \), where \( r \in k^* \) and \( \alpha, \beta \in k \), such that \( \alpha^2 - a\beta^2 = ab \).

We find the obstructions to the embedding problems related to group extensions \((3.4a)-(3.4d)\) in all five possible cases.

1. \( i \in k \). Then \( \xi \in k \), so \( \sigma \xi = \tau \xi = \xi, \chi'' = \chi \), and \( \chi' = \chi^{-1} \). Hence \( F_\chi = \langle \sigma, \tau \rangle \) and \( K_\chi = k(\sqrt{b}) \). By Theorem 2.1, the embedding problems related to group extensions \((3.4a)-(3.4d)\) are solvable \(\iff\) the embedding problem given by \( K/k(\sqrt{b}) \) and \((3.2)\), respectively, by \( K/k \) and

\[ 1 \to C_2 \to D_{2^a} \cong G/C_{2^{a+1}} \to D_8 \to 1 \]

are solvable. Here \( K/k(\sqrt{b}) = k(\sqrt{\alpha^2})/k(\sqrt{b}) \) for \( a' = [2r(\beta - i\sqrt{b})]^2 \). By Theorem 3.1 and Lemma 3.3, the obstructions for each embedding problem are \( (ab, 2)(-b, ra) \in Br(k) \) and \( (a, a')(\xi, \alpha' \beta') \in Br(k(\sqrt{b})) \), where we must have \( a' \in k(\sqrt{b})^* \), and \( b' \in k(\sqrt{b}) \) such that \( a'^2 - a\beta'^2 = \xi \).

2. \( a = -1 \). Then \( \sigma \xi = \xi^{-1}, \tau \xi = \xi, \) and \( \chi'' = \chi' = \chi^{-1} \). Hence \( F_\chi = \langle \sigma \rangle \) and \( K_\chi = k(i\sqrt{b}) \). The embedding problem \( K/k(\sqrt{b}) \) is solvable \(\iff\) the embedding problems \((K/k(i\sqrt{b}), \pi^{-1}C_2, \mu_{2^a})\) and \( (K/k, D_{16}, \mu_2) \) are solvable. Then we must have \( (-b, 2ar) = 1 \in Br(k) \), and the obstructions for each embedding problem are obtained as follows:

\[(3.4a), \pi^{-1}C_2^2 \cong D_{2^{a+2}}: \text{The embedding problem } (k(\sqrt{a^2}, i)/k(i\sqrt{b}), D_{2^{a+1}}, \mu_{2^a}) \text{ for } a' = (\phi + \psi)^2 = 2r(\alpha + i\sqrt{b}) \text{ is solvable }\iff\text{ the embedding problem } (k(\sqrt{a^2}, i)/k(i\sqrt{b}), D_{2^{a+1}}, \mu_{2^{a+1}}) \text{ is solvable for some } a'' = r^2a', \text{ and } r' \in k(i\sqrt{b}). \text{ Thus the obstruction to the embedding problem } (K/k, D_{2^{a+1}}, C_{2^{a}}) \text{ is } (a'', 2 - \xi - \xi^{-1}) = (2r(\alpha + i\sqrt{b}), 2 - \xi - \xi^{-1}) \in Br(k(i\sqrt{b})). \]

\[(3.4b), \pi^{-1}C_2^2 \cong Q_{2^{a+2}}: \text{The embedding problem } (k(\sqrt{a^2}, i)/k(i\sqrt{b}), Q_{2^{a+2}}, \mu_{2^a}) \text{ for } a' = 2r(\alpha + i\sqrt{b}) \text{ is solvable }\iff\text{ the embedding problem } (k(\sqrt{a^2}, i)/k(i\sqrt{b}), Q_{2^{a+1}}, \mu_{2^{a+1}}) \text{ is solvable for some } a'' = r^2a', \text{ and } r' \in k(i\sqrt{b}). \text{ Thus the obstruction to the embedding problem } (K/k, Q_{2^{a+1}}, C_{2^{a}}) \text{ is } (-1, -1)(2r(\alpha + i\sqrt{b}), 2 - \xi - \xi^{-1}) \in Br(k(i\sqrt{b})). \]

\[(3.4c), \pi^{-1}C_2^2 \cong Q_{2^{a+2}}: \text{The obstruction to the embedding problem } (K/k, SD_{2^{a+1}}, C_{2^{a}}) \text{ is } (-1, -1)(2r(\alpha + i\sqrt{b}), 2 - \xi - \xi^{-1}) \in Br(k(i\sqrt{b})). \]

\[(3.4d), \pi^{-1}C_2^2 \cong D_{2^{a+1}}: \text{The obstruction to the embedding problem } (K/k, SD_{2^{a+1}}, C_{2^{a}}) \text{ is } (2r(\alpha + i\sqrt{b}), 2 - \xi - \xi^{-1}) \in Br(k(i\sqrt{b})). \]
3. $b = -1$. This is the case considered in Theorem 3.2. We may assume that $r = \beta = 1$ and $\alpha = 0$. We must have $\alpha_2 \in k^*$, and $\beta_1 \in k$ such that $\alpha_2^2 + a\beta_1^2 = 2 - \zeta - \zeta^{-1}$. Then the obstructions are

\[(3.4a): (2 + \zeta + \zeta^{-1}, \alpha_1\beta_1)(a, \alpha_1(2\alpha_1 - \frac{\zeta - \zeta^{-1}}{i})) \in \text{Br}(k) \]
\[(3.4b): (-1, -1)(2 + \zeta + \zeta^{-1}, \alpha_1\beta_1)(a, \alpha_1(2\alpha_1 - \frac{\zeta - \zeta^{-1}}{i})) \in \text{Br}(k) \]
\[(3.4c): (2 + \zeta + \zeta^{-1}, \alpha_1\beta_1)(a, -\alpha_1(2\alpha_1 - \frac{\zeta - \zeta^{-1}}{i})) \in \text{Br}(k) \]
\[(3.4d): (-1, -1)(2 + \zeta + \zeta^{-1}, \alpha_1\beta_1)(a, -\alpha_1(2\alpha_1 - \frac{\zeta - \zeta^{-1}}{i})) \in \text{Br}(k) \]

4. $ab = -1$. Then $\alpha \zeta = \zeta^{-1}$, $\tau \zeta = \zeta^{-1}$, $\chi^3 = \chi^{-1}$, and $\chi^7 = \chi$. Hence $F_3 = \langle \alpha^2, \tau \rangle \cong C_2 \times C_2$ and $K_3 = k(\sqrt{a})$. The embedding problem $(K/k, G, C_2^e)$ is solvable $\iff$ the embedding problems $(K/k(\sqrt{a}), \pi^{-1}C_2^e, \mu_{2^e})$ and $(K/k, D_{16}, \mu_2)$ are solvable. Then we must have $(-b, or) = 1 \in \text{Br}(k)$, and the obstructions for each embedding problem are obtained as before:

\[(3.4a): \pi^{-1}C_2^e \cong D_{2+2};\text{ The obstruction to the embedding problem } (K/k, D_{2+2}, C_2^e) \text{ is } (r(\alpha + \beta\sqrt{a}), 2 - \zeta - \zeta^{-1}) \in \text{Br}(k(\sqrt{a})). \]
\[(3.4b): \pi^{-1}C_2^e \cong Q_{2+2};\text{ The obstruction to the embedding problem } (K/k, Q_{2+2}, C_2^e) \text{ is } (-1, -1)(r(\alpha + \beta\sqrt{a}), 2 - \zeta - \zeta^{-1}) \in \text{Br}(k(\sqrt{a})). \]
\[(3.4c): \pi^{-1}C_2^e \cong D_{2+2};\text{ The obstruction to the embedding problem } (K/k, D_{2+2}, C_2^e) \text{ is } (r(\alpha + \beta\sqrt{a}), 2 - \zeta - \zeta^{-1}) \in \text{Br}(k(\sqrt{a})). \]
\[(3.4d): \pi^{-1}C_2^e \cong Q_{2+2};\text{ The obstruction to the embedding problem } (K/k, D_{2+2}, C_2^e) \text{ is } (-1, -1)(r(\alpha + \beta\sqrt{a}), 2 - \zeta - \zeta^{-1}) \in \text{Br}(k(\sqrt{a})). \]

5. $a, b$ and $-1$ are quadratically independent. Let $\kappa$ generate $\text{Gal}(K(i)/K)$, and identify $\text{Gal}(K/k)$ with $\text{Gal}(K(i)/k(i))$. Then the embedding problem $(K/k, G, C_2^e)$ is solvable $\iff$ the embedding problem given by $K(i)/k(i)$ and

$$1 \to C_2^e \to G \times C_2 \to D_8 \times C_2 \to 1$$

is solvable. Here $(D_8 \times C_2)_\chi = \langle \sigma, \tau \kappa \rangle \cong D_8$ and $K(i)_\chi = k(i\sqrt{b})$. The restricted embedding problem is then given by $K(i)/k(i)$ and

$$1 \to \mu_{2^e} \to G \to D_8 \to 1.$$
Now let $k(\sqrt{a}, \sqrt{b})/k$ be a $C_2^2$ extension generated by elements $\rho_1$ and $\rho_2$ such that

$$
\rho_1: \sqrt{a} \mapsto -\sqrt{a}, \sqrt{b} \mapsto \sqrt{b}; \quad \rho_2: \sqrt{a} \mapsto \sqrt{a}, \sqrt{b} \mapsto -\sqrt{b}.
$$

Consider the embedding problem given by $k(\sqrt{a}, \sqrt{b})/k$ and

$$(3.5) \quad 1 \rightarrow C_{2n+1} \rightarrow G \rightarrow \frac{C_2^2}{n^p_1} \rightarrow 1,$$

where the group $G$ is generated by elements $x$ and $y$ such that $x$ is of order $2n+2$, $y^2 = x^{2n+1}$ or $y^2 = 1$, and $yx = x^{-1}y$ or $yx = x^{2n+1-1}y$. Hence $G$ is isomorphic to either $D_{2n+3}$, $Q_{2n+3}$, or $SD_{2n+3}$ groups. Obviously, this embedding problem is solvable if and only if $k(\sqrt{a}, \sqrt{b})/k$ can be embedded in a $D_8$ extension $K/k$ and the embedding problem $(K/k, G, C_{2^n})$ is solvable.

Again, extension (3.5) generates four group extensions:

$$(3.6a) \quad 1 \rightarrow C_{2n+1} \rightarrow D_{2n+1} \rightarrow \frac{C_2^2}{n^p_1} \rightarrow 1,$$

$$(3.6b) \quad 1 \rightarrow C_{2n+1} \rightarrow Q_{2n+1} \rightarrow \frac{C_2^2}{n^p_1} \rightarrow 1,$$

$$(3.6c) \quad 1 \rightarrow C_{2n+1} \rightarrow SD_{2n+1} \rightarrow \frac{C_2^2}{n^p_1} \rightarrow 1,$$

$$(3.6d) \quad 1 \rightarrow C_{2n+1} \rightarrow SD_{2n+1} \rightarrow \frac{C_2^2}{n^p_1} \rightarrow 1.$$

We write down the obstructions to the Brauer problems for $b = -1$, related to extensions (3.6a)–(3.6d).

Let $\zeta$ be a primitive $2n+1$th root of unity ($n > 1$) such that $\zeta + \zeta^{-1} \in k$ and $i(\zeta - \zeta^{-1}) \in k$. We can let $\alpha_i = \frac{\zeta - \zeta^{-1}}{2}$, $\beta_i = 0 : \alpha_1^2 + a\beta_1^2 = (\frac{\zeta - \zeta^{-1}}{2})^2 = 2 - \zeta^2 - \zeta^{-2}$. Then the obstructions are

(3.6a): $(a, 2 - \zeta - \zeta^{-1}) \in \text{Br}(k)$

(3.6b): $(-1, -1)(a, 2 - \zeta - \zeta^{-1}) \in \text{Br}(k)$

(3.6c): $(a, -2 + \zeta + \zeta^{-1}) \in \text{Br}(k)$

(3.6d): $(-1, -1)(a, -2 + \zeta + \zeta^{-1}) \in \text{Br}(k)$.

We proceed by investigating the embedding problem given by $k(\sqrt{b})/k$ and

$$(3.7) \quad 1 \rightarrow C_{2n+1} \rightarrow G \rightarrow \frac{C_2^2}{n^p_1} \rightarrow 1,$$

where the group $G$ again is isomorphic to either the $D_{2n+3}$, the $Q_{2n+3}$, or the $SD_{2n+3}$ group. Obviously, this embedding problem is solvable if and only if there exists $a \in k$ such that $a$ and $b$ are quadratically independent and the embedding problem given by $k(\sqrt{a}, \sqrt{b})/k$ and (3.5) is solvable.
Let $\zeta$ be a primitive $2^{n+2}$th root of unity ($n > 1$), such that $\zeta + \zeta^{-1} \in k$ and $i(\zeta - \zeta^{-1}) \in k$, and let $|k/k^2| \geq 4$. Again, we write down the obstructions to the Brauer problems for $b = -1$:

(3.6a): We have $(a, 2 - \zeta^2 - \zeta^{-2}) = 1 \in \Br(k)$ for all $a \in k$ such that $a$ and $-1$ are quadratically independent. Therefore, there is no obstruction, and it is easy to see that $k(\sqrt[2n]{\alpha}, i)/k$ is a solution to the embedding problem $(k(i)/k, D_{2n+1}, \mu_{2n+2})$.

(3.6b): The obstruction is $(-1, -1) \in \Br(k)$. This is exactly the same result obtained in [MZ, Example 3.4].

(3.6c): $(a, -1) \in \Br(k)$

(3.6d): $(-a, -1) \in \Br(k)$

Thus the embedding problem $(k(i)/k, D_{2n+1}, \mu_{2n+2})$ is solvable $\iff |k/k^2| \geq 4; (k(i)/k, Q_{2n+1}, \mu_{2n+2})$ is solvable $\iff |k/k^2| \geq 4$ and $(-1, -1) \in \Br(k)$; and $(k(i)/k, SD_{2n+1}, \mu_{2n+2})$ is solvable in both cases $\iff |k/k^2| \geq 4$ and $k$ is not quadratically closed.

Note that all of the obstructions in this section hold for “proper” solutions (i.e., Galois extensions), since we have $\rank(G) = \rank(C_2) = \rank(D_4) = 2$.

Finally, for $\zeta = i$ we can consider the group extension

$$1 \to C_4 \to G \xrightarrow{\gamma} D_8 \to 1,$$

where $G$ is isomorphic to either the $D_{32}, SD_{32},$ or $Q_{32}$ group. Then the obstruction to the Brauer problem is

$$(-1, e_i)(2, \alpha_1 \beta_1)(a, e_2 \alpha_2(\alpha_1 - 1)) \in \Br(k),$$

where $\alpha_1 \in k^*$, and $\beta_1 \in k$, such that $\alpha_1^2 + a\beta_1^2 = 2$. This coincides with Ledet’s result in [Le2].

Again, the group extension (3.8) generates four extensions:

(3.9a) $1 \to C_4 \to D_{32} \xrightarrow{\gamma \to \gamma} D_8 \to 1$

(3.9b) $1 \to C_4 \to Q_{32} \xrightarrow{\gamma \to \gamma} D_8 \to 1$

(3.9c) $1 \to C_4 \to SD_{32} \xrightarrow{\gamma \to \gamma} D_8 \to 1$

(3.9d) $1 \to C_4 \to SD_{32} \xrightarrow{\gamma \to \gamma} D_8 \to 1$.

We conclude the paper with several examples on Brauer problems related to extensions (3.9a)–(3.9d) over the rational field.

**Example 3.1.** Consider the embedding problem $(Q(\sqrt[2n]{\alpha}, i)/Q, G, \mu_4)$. We put $\alpha_1 = \frac{3}{4}, \beta_1 = \frac{1}{4} : \alpha_1^2 + 2\beta_1^2 = 2$, so the obstruction is $(-1, e_1)(2, \frac{3}{4})$.
(2, e_2^{1/2}) = (−1, e_1) ∈ Br(ℚ). Therefore, the embedding problems given by (3.9a) and (3.9c) are solvable, but those given by (3.9b) and (3.9d) are not.

**Example 3.2.** Consider the embedding problem \( (ℚ(√7, i)/ℚ, G, μ_4) \). We put \( α_1 = β_1 = 1/2 : α_1^2 + 7β_1^2 = 2 \), so the obstruction is \( (−1, e_1)(2, 1/2)(7, −e_2^{1/2}) = (−1, e_1)(7, −e_2) \). The obstructions for each embedding problem are

(3.9a): \( (7, −1) \neq 1 \in Br(ℚ) \)
(3.9b): \( (−7, −1) \neq 1 \in Br(ℚ) \)
(3.9c): \( (−1, 1)(7, 1) = 1 \in Br(ℚ) \)
(3.9d): \( (−1, −1) \neq 1 \in Br(ℚ) \).

Therefore, the embedding problems given by (3.9a), (3.9b), and (3.9d) are not solvable, but the embedding problem given by (3.9c) is solvable.

Of course, for an arbitrary rational number \( a \), it is very hard to determine whether the product of these three quaternion algebras is split in \( Br(ℚ) \). Computer-assisted calculations give the following example, where the embedding problem given by (3.9a) is solvable but the other embedding problems are not.

**Example 3.3.** Consider the embedding problem \( (ℚ(√−278877, i)/ℚ, G, μ_4) \). We put \( α_1 = 167, β_1 = 1 : α_1^2 − 27887β_1^2 = 2 \). Using the technique developed in [Mil], we can link the splitting of a quaternion algebra in \( Br(ℚ) \) to Legendre symbols. Since \( (√{278877})_2 = 1 \), we get \( (α_1β_1, 2) = (167, 2) = 1 \in Br(ℚ) \). We have \( −278877 = −79 ∙ 353 \) and \( α_1(α_1 − 1) = 2 ∙ 83 ∙ 167 \), so the obstruction is \( (−1, e_1)(−79 ∙ 353, e_2^{2} ∙ 83 ∙ 167) ∈ Br(ℚ) \). Note that \( 167 \equiv 7 (8), 79 \equiv 7 (8), \) and \( 353 \equiv 1 (8) \). Now \( (√{7})_2 = (√{3})_2 = 1 \), hence \( (−79 ∙ 353, 2) = 1; (√{83})_2 = (√{167})_2 = 1 \) and \( (√{79})_2 = (√{353})_2 = 1 \), hence \( (−79, 83 ∙ 167) = 1 \). Finally, \( (√{167})_2 = (√{353})_2 = 1 \), hence \( 353, 167 = 1 \), and \( (√{83})_2 = (√{353})_2 = 1 \), hence \( 353, 83 = 1 \).

Thus, if \( e_2 = 1 \), then the obstruction is \( (−1, e_1)(−79 ∙ 353, 2 ∙ 83 ∙ 167) = (−1, e_1)(353, 83 ∙ 167) = (−1, e_1)(353, 83) = (−1, e_1) = 1 ∊ Br(ℚ) ⇔ e_1 = 1 \). If \( e_2 = −1 \) and we assume that \( (−1, e_1)(−79 ∙ 353, −2 ∙ 83 ∙ 167) = 1 ∊ Br(ℚ) \), then in particular \( 79 ∙ 353 \) is a sum of three integer squares, which is an impossibility since \( 79 ∙ 353 \equiv 7(8) \). Therefore, we obtain that the embedding problem \( (ℚ(√−278877, i)/ℚ, D_{32}, μ_4) \) is solvable, but the other embedding problems are not.

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